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Real normalized differentials and compact cycles in the moduli space of curves

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Abstract

Using constructions of the Whitham perturbation theory of integrable system we prove a new sharp upper bound of $\lfloor \frac{3g}{2} \rfloor - 2$ on the dimension of complete subvarieties of $\mathcal{M}_g^{ct}$.

1 Introduction

Widely accepted by experts, but still conjectural, a "geometric explanation" for curious vanishing properties of the moduli space $\mathcal{M}_{g,k}$ of smooth genus g algebraic curves with punctures is the existence of its stratification by a certain number of affine strata, or the existence of a cover of $\mathcal{M}_{g,k}$ by a certain number of open affine sets (see [27] and references therein).

Historically, Arbarello first realized that a stratification of $\mathcal{M}_g$ could be useful for a study of its geometrical properties. He studied the stratification (known already to Rauch) $\mathcal{W}_2 \subset \mathcal{W}_3 \subset \cdots \subset \mathcal{W}_{g-1} \subset \mathcal{W}_g = \mathcal{M}_g$, where $\mathcal{W}_n$ is the locus of curves having a Weierstrass point of order at most n (i.e. the locus of curves on which there exists a meromorphic function with one pole of order at most n), and then conjectured that *any compact complex cycle in $\mathcal{M}_g$ of dimension $g - n$ must intersect $\mathcal{W}_n$*. Since $\mathcal{W}_2$ is the locus of hyperelliptic curves, which is affine, Arbarello's conjecture would imply that: *$\mathcal{M}_g$ does not contain complete (complex) subvarieties of dimension greater than $g - 2$*. This statement was later proved by Diaz in [4] with the help of a variant of Arbarello's stratification. Another modification of Arbarello's stratification was used by Loojenga, who proved that *the tautological classes of degree greater than $g - 2 + k$ vanish in the Chow ring of $\mathcal{M}_{g,k}$*, and then conjectured that $\mathcal{M}_{g,k}$ has a stratification with $g - \delta_{k,o}$ affine strata. His conjecture would be a consequence of the existence of a Zariski open cover of $\mathcal{M}_{g,k}$ made by $g - \delta_{k,0}$ affines. The existence of such covers is conjectured by Hain and Looijenga [15]; for genus up to 5 affine stratifications with the right number of strata were recently constructed by Fontanari and Looijenga [7],

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and covers — by Fontanari and Pascolutti [9], but no affine stratificaion, and not even a conjectural cover, is known beyond genus 5.

In [13, 14] the author jointly with S. Grushevsky proposed an alternative approach for geometrical explanation of the vanishing properties of $\mathcal{M}_{g,k}$ motivated by certain constructions of the Whitham perturbation theory of integrable systems [18, 19], further developed and clarified in [23, 24]. These constructions have already found applications in topological quantum field theories (WDVV equations) and $N = 2$ supersymmetric gauge theories [11] (see also [3] and references therein). The key elements of the alternative geometrical explanation are:

(a) the moduli space $\mathcal{M}_{g,k}^{(n)}$, $n = (n_1, \ldots, n_k)$ of smooth genus g Riemann surfaces Γ with the fixed n_α-jets of local coordinates in the neighborhoods of labeled points $p_\alpha \in \Gamma$ is the total space of a real-analytic foliation, whose leaves Ł are locally smooth complex subvarieties of real codimension $2g$;

(b) on $\mathcal{M}_{g,k}^{(n)}$ there is an ordered set of $(\dim_{\mathbb{R}}$ Ł$)$ continuous functions, which restrict to piecewise harmonic functions on the leaves of the foliation. Moreover, the first of these functions restricted onto Ł is a subharmonic function, i.e. it has no local maximum on Ł unless it is constant (if it is a constant then the next function is subharmonic, etc.).

The foliation structure arises through identification of $\mathcal{M}_{g,k}^{(n)}$ with the moduli space of curves with a fixed *real-normalized* meromorphic differential. By definition a real normalized meromorphic differential is a differential whose periods over any cycle on the curve are real. The power of this notion is that on any algebraic curve and for each fixed set of "singular parts" at the marked labeled points there exists a unique real normalized differential having prescribed singularities.

A new proof of Diaz' theorem, proposed in [13], uses the foliation structure defined by the real normalized differentials of the third kind, i.e. differentials with two simple poles. In [20] the arguments of this proof were extended for the case of real normalized differentials of the second kind (with no residues), and were used for the proof of Arbarello's conjecture, which had remained open until then. Highly non-trivial nature of the later problem has found its explanation in recent paper [2] by Arbarello and Mondello, where it was proved that $\mathcal{W}_n$ is almost never affine.

There are several partial compactifications of $\mathcal{M}_{g,n}$ for which analogs of vanishing results for the case of smooth curves are known (see for example [4, 8, 26, 27] and references therein). Possibly the most interesting among them is $\mathcal{M}_g^{ct}$, the moduli space of stable curves of compact type, i.e. those stable curves where the Jacobian is compact; equivalently, whose dual graph is a tree.

An easy corollary of Diaz' bound on the dimension of compact cycles in the moduli space of smooth curves is the statement (which is also due to Diaz [4]) that there is no compact cycle in $\mathcal{M}_g^{ct}$ of dimension greater that $2g - 3$. In fact, a stronger result is true. In [16] Keel and Sadun proved that for $g \geq 3$ there do not exist complete complex subvarieties of $\mathcal{M}_g^{ct}$ of dimension greater than $2g - 4$. The proof in [16] for arbitrary g is by easy induction arguments starting from the base case of $g = 3$. The proof of the base statement that there does not exist a compact threefold in $\mathcal{M}_3^{ct}$ is of a different nature. Keel and Sadun obtained

it as a corollary of their remarkable vanishing result: there does not exist a complete complex subvariety of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of codimension g.

The seemingly humble improvement in [16] of the previous bound is quite significant in relation to Faber's conjectures [6] or, more precisely, in relation to their analog for $\mathcal{M}_g^{ct}$ (see extensive discussion in [16]). In this connection, our main result looks even more striking:

Theorem 1.1. *For $g \geq 2$ there do not exist complete complex subvarieties of $\mathcal{M}_g^{ct}$ of dimension greater than $\lfloor \frac{3g}{2} \rfloor - 2$.*

It is known that $\mathcal{M}_2^{ct}$ contains a complete curve, and that $\mathcal{M}_3^{ct}$ contains a complete surface. Therefore, the bound above is sharp for $g = 2, 3$.

Formally it is sharp for all g because of the following trivial reason (pointed out to the author by Sam Grushevsky). The complete subvarieties of maximal dimension in the boundary of $\mathcal{M}_g^{ct}$ can be constructed explicitly. Let X_1, X_2 be complete subvarieties in $\mathcal{M}_i^{ct}$ and $\mathcal{M}_{g-i}^{ct}$, respectively. Choosing a point on each curve and attaching two curves at the corresponding points one gets a complete subvariety X of $\mathcal{M}_g^{ct}$. If $i = 1$, the dimension of this subvariety equals $\dim X_2 + 1$. For $1 < i < g - 1$ it is equal to $\dim X_1 + \dim X_2 + 2$. Gluing together n copies of a complete curve in $\mathcal{M}_2^{ct}$ one gets a complete cycle in $\mathcal{M}_{2n}^{ct}$ of dimension $3n - 2$. Gluing a complete surface in $\mathcal{M}_3^{ct}$ and $n - 1$ copies of a complete curve in $\mathcal{M}_2^{ct}$ one gets a complete cycle in $\mathcal{M}_{2n+1}^{ct}$ of dimension $3n - 1$.

A direct induction show that it is sufficient to prove the statement of the theorem under the assumption that $X \subset \mathcal{M}_g^{ct}$ has nontrivial intersection with $\mathcal{M}_g$, i.e. throughout the paper it is always assumed that the generic curve in X is smooth.

It is not clear to the author if the bound in Theorem 1.1 is sharp for $g \geq 4$ under the assumption that a generic curve in X is smooth. In fact, we conjecture a much stronger bound:

Conjecture 1.2. *Let X be a complete complex subvariety in $\mathcal{M}_g^{ct}$, having non-empty intersection with $\mathcal{M}_g$. Then for $g \geq 1$ it is of dimension at most $g - 1$.*

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2 Necessary facts and constructions

Let (Γ, p_-, p_+) be a smooth genus g algebraic curve with two marked points. Non-degeneracy of the imaginary part of the Riemann matrix (the matrix of B- periods of the basis of holomorphic differentials dual to an arbitrary chosen basis of cycles on Γ with the canonical intersection matrix) implies that on Γ there exists a unique meromorphic differential Ψ with residues $\mp i$ at simple poles at $p_\pm$ and having *real periods* over any cycle on Γ. Globally it can be seen as the real analytic section Ψ over $\mathcal{M}_{g,2}$ of the bundle of meromorphic differentials with two simple poles (with residues $\mp i$).

Remark 2.1. The real normalized differentials of the third kind per se are not new. They were probably known to Maxwell (the real part of such differential is a single valued harmonic function on Γ equal to the potential of electromagnetic filed on Γ created by charged particles at the marked points); they were used in the, so-called, light-cone string theory [10], and played a crucial role in joint works of S. Novikov and the author on Laurent-Fourier theory on Riemann surfaces and on operator quantization of bosonic strings [21, 22].

Identification of $\mathcal{M}_{g,2}$ with the moduli space of smooth Riemann surfaces Γ with fixed real normalized differential Ψ having two simples poles with residues $\mp i$ allows us to introduce on $\mathcal{M}_{g,2}$ a foliation structure.

Definition 2.2. A leaf Ł of the foliation on $\mathcal{M}_{g,2}$ is defined to be the locus along which the periods of the corresponding differential Ψ remain constant.

The foliation we define is real-analytic, but its leaves are complex (so the foliation is real-analytic "in the transverse direction"). Indeed, locally in the neighborhood U of a curve Γ_0 one can always choose a basis of cycles on every curve $\Gamma \in U$, which continuously varies with a variation of Γ. Therefore, locally a leaf Ł is defined to be the locus where the integrals of Ψ over the chosen basis of cycles $A_1, \ldots, A_g, B_1, \ldots, B_g$ are equal to $a_1, \ldots, a_g, b_1, \ldots b_g$ — these are holomorphic conditions, and thus the leaf is a complex submanifold Ł $\subset \mathcal{M}_{g,2}$. If a different basis of $H_1(\Gamma, \mathbb{Z})$ is chosen, the periods of Ψ over the basis are still fixed along a leaf (though numerically different).

It is a basic fact of the Whitham theory proved in full generality (for real normalized differentials having poles of arbitrary fixed order of poles at k punctures) in [23]

Theorem 2.3. *A leaf Ł is a smooth complex subvariety of real codimension* $2g$.

A set of holomorphic coordinates on Ł is similar to the ones in the theory of Hurwitz spaces. They are "critical" values of the corresponding abelian integral $F(p)$, $p \in \Gamma$

$$F(p) = c + \int^p \Psi, \tag{1}$$

which is a multivalued meromorphic function on Γ. The zero divisor of Ψ is of degree $2g$. At the generic point of Ł, where all the zeros q_s of Ψ are distinct, the coordinates on Ł are the values of F at these critical points:

$$\varphi_s = F(q_s), \quad \Psi(q_s) = 0, \quad s = 0, \ldots, 2g-1, \tag{2}$$

normalized by the condition $\sum_s \varphi_s = 0$. Of course, these coordinates depend upon the path of integration needed to define F in the neighborhood of q_s. The normalization above is needed to define in addition the common constant c in (1). At points of Ł where the corresponding differential has multiple zeros $q_{s_1} = \ldots = q_{s_r}$, the local coordinates are symmetric polynomials $\sigma_i(\varphi_{s_1}, \ldots, \varphi_{s_r})$, $i = 1, \ldots, r$ (it is assumed here that the paths of integration for critical values φ_{s_k} are chosen consistently; for more details see [23]).

A direct corollary of real normalization is the statement that imaginary parts $f_s = \Im\varphi_s$ of the critical values are independent of paths of integration, and depend only on labeling of the critical points. They can be arranged into decreasing order

$$f_0 \geq f_1 \geq \cdots \geq f_{d-1} \geq f_{2g-1}, \quad \sum_{j=0}^{2g-1} f_j = 0. \tag{3}$$

Then f_j is a well-defined continuous function on $\mathcal{M}_{g,2}$, which restricted onto Ł is a piecewise harmonic function. Moreover, as shown in [13], the first function f_0 restricted onto any leaf Ł is a *subharmonic function*, i.e. it is a function for which the maximum principle can be applied: *f_0 has no local maximum on Ł unless it is constant* (if f_0 is a constant, then f_1 is subharmonic, and so on).

Subfoliations. For further use let us introduce additional foliation structures on $\mathcal{M}_{g,2}$.

Definition 2.4. A leaf $l \subset$ Ł of the foliation on $\mathcal{M}_{g,2}$ is defined to be the locus along which the periods and a subset of marked n critical values $\varphi_{s_1}, \ldots, \varphi_{s_n}$ of the corresponding differential Ψ remain constant.

Of course, the numerical values φ_{s_j} depend upon a choice of a path of integration needed to define the abelian integral F in the neighborhood of q_{s_j}, but the condition that they remain constant along l is independent. Hence, locally l is a well-defined complex subvariety of real codimension $2g + 2n$ in $\mathcal{M}_{g,2}$, which at the same time can be seen as complex subvariety of complex codimension n of the corresponding leaf Ł of the big foliation. It is smooth when the corresponding zeros are simple, and singular otherwise.

3 Extension to the boundary

Our next goal is to extend the notion of real normalized differentials to the case of stable curves of compact type, and to describe the asymptotic behavior of their critical values near the boundary divisors.

The line bundle of meromorphic differentials with two simple poles, i.e. the bundle with fiber $K_\Gamma + p_+ + p_-$ over a smooth curve, extends to a bundle globally over the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,2}$ — the fiber over a stable curve Γ is $\omega_\Gamma + p_+ + p_-$, where ω_Γ is the relative dualizing sheaf. Recalling the definition of the relative dualizing sheaf, analytically this means that limits of meromorphic differentials with prescribed poles at p_i are meromorphic differentials with poles at the points $p_\pm$ and possibly with simple poles at the nodes, with the residues from the two components canceling.

Remark 3.1. In general if one takes a family of meromorphic differentials on smooth Riemann surfaces (i.e. takes a section over $\mathcal{M}_{g,n}$), we expect that the limit may have simple poles at the nodes. Moreover, the theory of limit linear series on reducible curves is extremely complicated, see for example [5], and to determine all possible limits of sections on reducible nodal curves, one may need to twist the bundle by some multiples of the connected

components of the nodal curve. It turns out that this does not happen for the differentials of the second kind with real periods. More precisely, in [13] it was shown that the limit of the real normalized differentials having one pole of the second order at a marked point p_0 at a stable curve Γ is the unique meromorphic differential that is identically zero on all connected components of the normalization $\tilde{\Gamma}$ (geometrically $\tilde{\Gamma}$ is obtained from Γ by detaching the attached nodes) except the one containing p_0. On that component Ψ_Γ is the unique differential with real periods and prescribed singular part at the double pole at p_0.

In full generality limits of real normalized differentials (having poles of arbitrary orders at several points and non-zero residues) at reducible stable curves are not well understood yet, and deserve a more systematic study. For our further purposes, we need only to consider limits of real normalized differentials with two simple poles at stable curves of compact type, which admits a simple description.

Let $\mathcal{M}_{g,2}^{ct}$ be the moduli space of stable curves of compact type with two labeled points $p_\pm$. Throughout the paper a pair of labeled points on a stable curve of compact type such that tthe preimages of these points under the normalization map are on the same irreducible component of the normalization is called an *irreducible pair of labeled points.* (a pair of points on a smooth curve is always irreducible).

The dual graph of a stable curve Γ of compact type is a tree. Therefore, there exists a unique *oriented* path in the dual graph connecting the two irreducible components containing two marked points. (The positive orientation of the path is the orientation from p_- towards p_+). Let Γ_α, $\alpha = 1, \ldots, k$, be connected components of the normalization $\widetilde{\Gamma}$, corresponding to vertices of that path, ordered according to this orientation. On each of the curves Γ_α there are two points $p_\pm^\alpha$ that are preimages of nodes or the initial marked points $p_\pm$. More precisely, on Γ_1 the two parked points are the first initially marked point $p_-^1 = p_-$ and the preimage p_+^1 of the node connecting Γ_1 with Γ_2. The preimage of the same node on Γ_2 is the marked point p_-^2. Continuing this labeling of points, we end up by identifying p_+^k with p_+.

Let $\mathcal{M}_{g,2}^{ct,(k)}$ be the locus where the path in the dual graph connecting the irreducible components containing p_+ and p_- has k vertices. Detaching the nodes corresponding to edges of the path one gets an ordered set of stable curves with irreducible pairs of labeled points, i.e.

$$\mathcal{M}_{g,2}^{ct,(k)} = \prod_{\alpha=1}^{k} \mathcal{M}_{g_\alpha,2}^{ct,(1)}. \tag{4}$$

(the product at the right hand side is ordered).

Lemma 3.2. *The real analytic section Ψ over $\mathcal{M}_{g,2}$ of the bundle of meromorphic differentials with two simple poles (with residues $\mp i$) extends to a real analytic section of the extension of this bundle, $\omega_\Gamma + p_- + p_+$ over $\mathcal{M}_{g,2}^{ct}$. For a stable curve $(\Gamma, p_\pm)$ the section Ψ_Γ is the unique meromorphic differential that is identically zero on all connected components of the normalization $\tilde{\Gamma}$ except at the chain of components Γ_j connecting $p_\pm$. On Γ_j Ψ_Γ is the unique real normalized differential with simple poles at $p_\pm^\alpha$ and residues $\mp i$.*

For the proof it is enough to consider the closure of the section Ψ in the total space of the bundle $\omega_\Gamma + p_- + p_+$ over $\mathcal{M}_{g,2}^{ct}$, and notice that on connected components of $\widetilde{\Gamma}$ the real

normalization uniquely defines preimages of limiting differential Ψ_Γ under the normalization map. Real analyticity of the extended section is a direct corollary of the fact that Torelli map (Riemann matrix of B-periods of normalized holomorphic differentials) extends analytically onto $\mathcal{M}_g^{ct}$.

From the analyticity of the Torelli map it follows that periods of the real normalized differentials Ψ on smooth curves, regarded as locally defined functions on $\mathcal{M}_{g,2}$, extend analytically to $\mathcal{M}_{g,2}^{ct}$, as "periods" of the section of $\omega_\Gamma + p_+ + p_-$, but the differentials of the periods are linearly independent only on part of the boundary of $\mathcal{M}_{g,2}^{ct}$. Namely, they are linearly independent on the divisors corresponding to singular curves whose dual graph is just the path connecting the components of $\widetilde{\Gamma}$ containing marked points (i.e. $\widetilde{\Gamma}$ is a "chain" of curves Γ_α in Lemma 3.2, and does not contain components on which the limit of Ψ is identically zero). Therefore, the foliation structure on $\mathcal{M}_{g,2}$ extends smoothly only through the later part (called below the regular part) of the boundary.

Lemma 3.2 implies that the functions f_j on $\mathcal{M}_{g,2}$ (imaginary parts of the critical values) extend as continuous functions through the part of the boundary corresponding to singular curves with an irreducible pair of labeled points. (Recall, that the later means that the preimages of these points under normalization belong to the same connected component Γ_0). In the neighborhood of such a point of the boundary the $2g$-tuple of functions f_j splits into a set of $2g_s$-tuples corresponding to each irreducible component of the normalization. The first $2g_0$-tuple corresponding to the principal component containing the marked points, and is of the form $f_{i,0}+c$, where $f_{i,0}$ in the limit tend to the imaginary parts of the critical values Ψ_0 on Γ_0. At the boundary the values of functions in each of the other tuples coincide and are equal to $\Phi_0(q_s)+c$, where Φ_0 is the imaginary part of F_0, normalized by the condition $\sum_j f_{j,0} = 0$, and q_s is the preimage of the node connecting components Γ_0 and Γ_s. The constant c is defined by the normalization condition on the singular curve Γ:

$$gc + \sum_s g_s \Phi_0(q_s) = 0. \tag{5}$$

In general the functions f_j have no finite limit at boundary points where the preimages of marked points belong to different irreducible components of the normalization. An explicit description of the asymptotic behavior of f_j under corresponding degenerations is revealed by the following gluing construction.

Consider first degenerations to a singular curve with two labeled points such that the path in the dual tree connecting irreducible components with preimages of the labeled points contains only two vertices (the case $k = 2$ above).

Let Γ_1, Γ_2 be stable curves of compact type with fixed irreducible pairs of points $p_\pm^1$ and $p_\pm^2$, respectively, and let Ψ_1 and Ψ_2 be the corresponding real normalized differentials. In the neighborhood of p_+^1, where Ψ_1 has residue $-i$, the imaginary part Φ_1 of the abelian integral F_1 tends to $+\infty$. In the neighborhood of p_-^2 the imaginary part Φ_2 of the abelian integral F_2 tends to $-\infty$. Let us fix a complex parameter t and define the neighborhood $U_1^t \subset \Gamma_1$ of p_+^1 and the neighborhood $U_2^t \subset \Gamma_2$ of p_-^2 by the inequalities:

$$q_1 \in U_1^t \mid \Phi_1(q_1) > -\frac{1}{2}\ln|t|\,; \quad q_2 \in U_2^t \mid \Phi_2(q_2) < \frac{1}{2}\ln|t| \tag{6}$$

If $|t|$ is sufficiently small, then the boundaries ∂U_i^t are circles which can be identified via the implicit equation

$$F_2(q_2) = F_1(q_1) + \ln t, \quad q_1 \in S^1 = \partial U_1^t, \; q_2 \in S^1 = \partial U_2^t. \tag{7}$$

Using this identification we define a new stable algebraic curve of compact type with an irreducible pair of labeled points, first, topologically gluing the complements $\Gamma_i \setminus U_i^t$ along the boundary circles. The complex structure on $\Gamma_t = (\Gamma_1 \setminus U_1^t) \cup_{S^1} (\Gamma_2 \setminus U_2^t)$ is defined in a conventional way: locally holomorphic functions in the neighborhood of a point on the circle S^1 are *continuous* functions that are holomorphic outside of the circle (with respect to complex structures on Γ_i). The differential Ψ_t that is equal to Ψ_1 on $\Gamma_1 \setminus U_1^t$ and Ψ_2 on $\Gamma_2 \setminus U_2^t$ is continuous across S^1, therefore, it is the real normalized differential on Γ_t with simple poles at $p_- = p_-^1$ and $p_+ = p_+^2$. As $t \to 0$, the curve Γ_t degenerates to the singular curve Γ_0 with irreducible components Γ_i.

The zeros of Ψ_t are the zeros of Ψ_1 on $(\Gamma_1 \setminus U_1^t)$ and the zeros of Ψ_2 on $(\Gamma_2 \setminus U_2^t)$. By continuity, a branch $F_1 + c$ of the abelian integral F_t of Ψ_t on the first part of Γ_t extends as $F_2 - \ln t + c$. If the abelian integrals F_1, F_2 are normalized so that the imaginary parts $f_{j,1}$ and $f_{j,2}$ of their critical values satisfy the conditions $\sum_j f_{j,1} = \sum_j f_{j,2} = 0$, then the normalization condition for critical values of F_t defines the constant c above: $(g_1 + g_2)\, c = g_2 \ln t$. Hence, the tuple of functions f_j in the neighborhood of the singular curve Γ_0 splits into two tuples of the form

$$f_{j,1} + \frac{g_2}{g} \ln |t|\,; \quad f_{j,2} - \frac{g_1}{g} \ln |t|\,, \quad g = g_1 + g_2. \tag{8}$$

Similarly, taking a set $(\Gamma_\alpha, p_\pm^\alpha)$, $\alpha = 1, \ldots, k$, of stable curves of compact type with irreducible pairs of labeled points and taking a set of sufficiently small complex parameters $t = (t_1, \ldots, t_{k-1})$ one can construct a stable curve Γ_t with an irreducible pair of marked points. The corresponding $(k-1)$-parametric family at $t = (0, \ldots, 0)$ degenerates to the singular curve Γ_0. Therefore, in the neighborhood of a point of the stratum $\mathcal{M}_{g,2}^{ct,(k)}$ the tuple of functions f_j splits into k tuples of function having the form

$$f_{j,\alpha} + c - \sum_{s=1}^{\alpha-1} \ln |t_s|, \quad \sum_j f_{j,\alpha} = 0\,. \tag{9}$$

The constant c, defined by the normalization condition, is given by

$$c = \frac{1}{g} \sum_{s=1}^{k-1} \Big(\sum_{\alpha < s} g_\alpha\Big) \ln |t_s|, \quad g = \sum_{\alpha=1}^{k} g_\alpha. \tag{10}$$

4 Proof of the main theorem.

For motivation of further arguments, let us briefly outline the key steps of the proof of Diaz' theorem in [13]. The fiber of the forgetful map $\mathcal{M}_{g,2} \to \mathcal{M}_g$ over the point Γ is $\Gamma \times \Gamma \setminus$ diagonal, and thus non-compact. A partial compactification of $\mathcal{M}_{g,2}$ is the square of the universal curve $\mathcal{C}_g^2$ which is the moduli space of curves with two labeled points (not

necessarily distinct). The fiber of the map $\mathcal{C}_g^2 \to \mathcal{M}_g$ over Γ is $\Gamma \times \Gamma$, and thus is compact. From the point of view of the Deligne-Mumford compactification, if the two marked points coincide, we attach a nodal $\mathbb{CP}^1$ at this point. The real normalized differential defined on $\mathcal{M}_{g,2}$ has zero limit on the diagonal $\mathcal{C}_g = \mathcal{M}_{g,1} \subset C_g^2$. Indeed, if the points $p_\pm$ coincide, then the corresponding differential Ψ becomes a *holomorphic* differential. Since in the limit Ψ is still real-normalized, in the limit it becomes identically zero on Γ, and the associated functions f_j all become zero.

Let X be a compact cycle in $\mathcal{M}_g$. Its preimage Z under the forgetful map $\mathcal{C}_g^2 \longmapsto \mathcal{M}_g$is compact. The function f_0, as a continuous function, achieves its supremum on Z. It is easy to see that the function f_0 is not identically zero on the preimage Y of X under the forgetful map $\mathcal{M}_{g,2} \to \mathcal{M}_g$, $Y \subset Z \subset \mathcal{C}_g^2$ (see details in [13]). Thus f_0 achieves its supremum at a point in Y (Recall that $f_0 \equiv 0$ on the diagonal $\mathcal{C}_g$). Let Ł be the leaf of the big foliation passing through that point. At this point the function f_0 restricted onto Ł $\cap\, Y$ has a local maximum. Then, it must be a constant on Ł $\cap\, Y$.

Let $Y_0 \subset Y$ be the compact set, where f_0 takes its maximum value. On the compact set the second function f_1 must achieve its supremum. The above discussion shows that Y_0 is foliated by leaves Ł $\cap\, Y$ (i.e. that for any leaf Ł intersecting Y_0, Ł $\cap\, Y =$ Ł $\cap\, Y_0$). On these leaves the second function f_1 is subharmonic, i.e. it must be a constant. Continuing the induction step we get that all functions f_s are constants on Ł $\cap\, Y$. If $f_s = \Im\varphi_s$ is constant, then φ_s is also (locally) constant on Ł $\cap\, Y$. Since the functions φ_s are local coordinates on Ł, Ł $\cap\, Y$ must be zero-dimensional. However, if X is of dimension greater that $g-2$, then Ł $\cap\, Y$ is at least one-dimensional. The contradiction completes the proof of Diaz' theorem.

Consider now a complete cycle $X \subset \mathcal{M}_g^{ct}$ such that its intersection $X^0 = X \cap \mathcal{M}_g$ is not empty, i.e. the generic curve in X is smooth. For a fixed non-zero real number a the locus Y_a defined to be the union of level sets $\{Y_{a,j} \mid f_j = a\}$ of the functions f_j restricted onto the preimage $Y \subset \mathcal{M}_{g,2}$ of X^0 under the forgetful map, i.e. $Y_a = \cup_j Y_{a,j}$. In other words, Y_a is the locus where at least one of the functions f_j takes value a. Similarly, a joint level Y_A, $A = (a_1 < a_2 < \ldots < a_n)$ is defined to be the locus where a set of n functions f_j take fixed values, i.e. $Y_A = \cap_{i=1}^n Y_{a_i}$.

Lemma 4.1. *If X is of dimension at least $g+n-1$, then the closure $\overline{Y}_A$ of the joint level set Y_A, $(|A| = n)$ in $\mathcal{M}_{g,2}^{ct}$ intersects the boundary, i.e. $\overline{Y}_A \cap \partial\mathcal{M}_{g,2}^{ct} \neq \emptyset \subset \mathcal{M}_{g,2}^{ct}$.*

Proof. Suppose that the closure of Y_A does not intersect the boundary. Then Y_A is compact. On a compact set the continuous function f_0 achieves its maximum at some point $(\Gamma, p_\pm)$. By definition of Y_A at this point $f_{j_i} = a_i$ for at least one j_i. Consider the leaf λ of the subfoliation passing through $(\Gamma, p_\pm)$ along which the periods and the corresponding critical values ϕ_{j_i}, $\Im\phi_{j_i} = a_i$, remain constant. If the zero q_{j_i} is simple, then λ is smooth at $(\Gamma, p_\pm)$. The function f_0 has local maximum at this point on $\lambda \cap Y$. Hence, f_0 is constant on $\lambda \cap Y$. Continuing arguments along the same line as in the proof of Diaz's theorem above, one gets that all the other functions f_j are constant on $\lambda \cap Y$, as well. Hence, $\lambda \cap Y$ is zero-dimensional. But if X is of dimension at least $g+n-1$, then $\lambda \cap Y$ is at least one-dimensional. Thus we have arrived at a contradiction.

Only a slight modification of the previous arguments is needed in the case when the zeros q_{j_i} of the differential Ψ are not simple at the point where f_0 achieves its maximum on Y_A. (These arguments are similar to the ones already used in analogous situation in [13].) In the neighborhood of that point the set q_{j,s_i} of (unlabeled) zeroes that become the multiple zero in the limit varies holomorphically. Therefore, locally the functions $\phi_{j_i} = \sum_s \varphi_{j,s_i}$ are holomorphic on Ł. Locally the level sets l' of these function are smooth, and can be used instead of l in the argument above. □

The following statement is at the heart of the rest of the proof. Its geometric meaning seems highly non-trivial: if a complete cycle $X \subset \mathcal{M}_g^{ct}$ is of dimension at least g, then there is an irreducible component of the intersection $X \cap \partial\mathcal{M}_g^{ct}$ which is contained in the locus of singular curves whose normalization has *at least three* connected components. Notice that the locus of such stable curves is of codimension 2 in $\mathcal{M}_g^{ct}$, while a connected component of the intersection of X with boundary is of codimension 1 in X.

Recall that $\mathcal{M}_{g,2}^{ct,(k)}$ is the locus where the path in the dual graph connecting irreducible components containing the labeled points has k vertices.

Lemma 4.2. *The locus $Y_A^\infty \subset \partial\mathcal{M}_{g,2}^{ct}$ of limiting points of the joint level set Y_A, as $f_0 \to \infty$, does not intersect the strata $\mathcal{M}_{g,2}^{ct,(k)}$ with $k = 1, 2$.*

Proof. The proof of the lemma is an easy corollary of the description of asymptotic behavior of the critical values near the boundary. Equation (5) implies that in the neighborhood of any point of $\mathcal{M}_{g,2}^{ct,(1)}$ the function f_0 is bounded. Therefore, $Y_A^\infty \cap \mathcal{M}_{g,2}^{ct,(1)} = \emptyset$. The arguments showing that $Y_A^\infty \cap \mathcal{M}_{g,2}^{ct,(2)} = \emptyset$ are just opposite: from (8) it follows that *none of* the functions f_j have a finite limit at points of the strata $\mathcal{M}_{g,2}^{ct,(2)}$, i.e. none of the functions f_j takes prescribed value a near any point of the strata. □

The formulae (9) and (10) describe the asymptotic behavior of f_j in the universal family depending on $k-1$ complex parameters. The limiting behavior along a one-parametric family is given by the same formulae with t_s being holomorphic functions of the parameter t in the neighborhood of $t = 0$, i.e. in the leading order $\ln|\,t_s(t)| = \nu_s \ln|\,t|$, where ν_s are positive integers. Then, from (9) and (10) it follows that only one tuple (corresponding to one value of $1 < \alpha_0 < k$ in the decomposition (4)) of critical values functions might have a finite limit. All other critical values f_j tend to $\pm\infty$. Along each leaf Ł of the big foliation the degeneration with $f_0 \to \infty$ is a one-parametric complex family. Hence, each connected component of the locus Y_A^∞ is contained in the stratum of the boundary of the form

$$\mathcal{M}_{g_-,2}^{ct} \times \mathcal{M}_{g_0,2}^{ct,(1)} \times \mathcal{M}_{g_+,2}^{ct}, \quad g_- + g_0 + g_+ = g,\ g_\pm > 0,\ g_0 > 0,\ . \tag{11}$$

The first and the last factor of the decomposition (11) combine factors in (4) where the critical values tend to $-\infty$ or $+\infty$, respectively.

The locus Y_A^∞ is not complex, but the condition that under degeneration $f_0 \to \infty$, there is a subset of n critical values functions having finite (but not fixed) limits is open complex condition. Therefore, the locus $Y_n^\infty = \cup_{A,\ |A|=n} Y_A^\infty$ is complex of complex codimension 1 in $\overline{Y}$ (provided that it is not empty).

Now we are ready to complete the proof of the main theorem by induction. The base of the induction is the case $g = 2$. The corresponding statement that $\mathcal{M}_2^{ct}$ does not contain a complete surface is well-known. Nevertheless, it is instructive to get it as a direct corollary of the lemmas above.

Suppose that X is a complete surface in $\mathcal{M}_2^{ct}$. Then, by Lemma 4.1 the locus Y_a^∞ is not empty. On the other hand, by Lemma 4.2 it is empty, because $\mathcal{M}_{2,2}^{ct,(k)}$ with $k = 1, 2$ are the only stratum of the boundary.

Suppose now that X is a complete cycle in $\mathcal{M}_g^{ct}$ of dimension $g + n - 1$. Consider a connected component of the closure $\overline{Y_n^\infty}$ in one of the strata (11). Let Z_n be the image of $\overline{Y_n^\infty}$ under the projection (forgetful map) of the strata (11) onto the middle factor. It is a complete cycle in $\mathcal{M}_{g_0,2}^{ct}$. The intersection of its open part with leaves of the big foliation on $\mathcal{M}_{g_0,2}$ is at least n-dimensional (parameters a_i in the definition of level set above). This is impossible for $g_0 = 1$, while for $g_0 \geq 2$ by the induction assumption we have $2n \leq g_0 \leq (g-2)$. Therefore, $\dim X = g + n - 1 \leq \frac{3g}{2} - 2$. Thus, the main theorem is proved.

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